Set Simulations for Quadratic Systems

Giuseppe Calafiore

***Abstract*—In this note, we study the problem of propagating in time a bounding set for the state of a class of nonlinear quadratic systems. The sequence of bounding sets is called the *set simulation* of the system, and conveys useful information about the stability and qualitative behavior of the possible time responses of the system. Numerically efficient recursive algorithms are presented for the specific cases when the bounding sets are orthotopes or ellipsoids.**

***Index Terms*—Domain of attraction, nonlinear systems, quadratic systems, semidefinite relaxations, set simulations.**

I. Introduction and Problem Statement

The problem of determining geometrical regions in the state space that contain all possible reachable states of a dynamical system has been extensively studied in the literature. In the classical deterministic (or set-membership) filtering literature [3], [15], [22] an ellipsoidal bounding set for the state of a linear system is computed recursively, starting from deterministic assumptions on the noise affecting the system, which is assumed to be an unknown-but-bounded (ubb) sequence, instead of a stochastic sequence. In the same context, interval analysis is also used to propagate in time intervals of confidence for the states, and to update this information with upcoming measurements, [12]. For linear systems with uncertain parameters, a polytopic bounding approach was proposed in [1], while more recently an ellipsoidal bounding technique has been proposed in [8], [9]. Ellipsoids have also been used as target invariant sets in the context of model predictive control for uncertain systems in [5].

In this note, we propose numerically efficient algorithms for recursively determining orthotopic or ellipsoidal bounds for the state of a class of nonlinear quadratic, discrete-time systems. From a theoretical point of view, quadratic systems are an important class of nonlinear polynomial systems, and encompass the much studied class of bilinear systems [14], [17]. Moreover, quadratic systems arise naturally in the context of generic nonlinear systems, when local analysis is to be performed using second order Taylor series approximation around an equilibrium.

Specifically, we here consider an autonomous discrete-time nonlinear system described by the state-difference equations

$$x(k+1) = f\left(x(k)\right) \tag{1}$$

where $x(k) \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that each component $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, is a quadratic function of x

$$f_i(x) = x^T Q_i x + 2 b_i^T x + c_i \tag{2}$$

where $Q_i = Q_i^T \in \mathbb{R}^{n,n}$, $b_i \in \mathbb{R}^n$, $c_i \in \mathbb{R}$ are given matrices. Suppose that $\hat{x}(0) \in \mathbb{R}^n$ is a nominal initial state for (1), and that $\hat{x}(k)$, $k = 0, 1, \ldots$ is the resulting nominal state trajectory obtained from the initial condition $x(0) = \hat{x}(0)$.

We consider the following problem. Assume that the initial state of the system, $x(0)$, is such that $x(0) - \hat{x}(0) \in \mathcal{X}(0)$, where $\mathcal{X}(0)$ is a given bounded subset of $\mathbb{R}^n$, and let $x(k)$ be the resulting state trajectory. Our goal is to determine a sequence of sets $\mathcal{X}(k)$, $k = 1, 2, \ldots$ of minimal "size" (in a sense to be clarified later on) such that $x(k) - \hat{x}(k) \in \mathcal{X}(k)$, $k = 1, 2, \ldots$, for any $x(0)$ such that $x(0) - \hat{x}(0) \in \mathcal{X}(0)$. In this work, the sets $\mathcal{X}(k)$ are assumed to be either orthotopes or ellipsoids, and the sequence $[\mathcal{X}(k)]_{k=1,2,\ldots}$ is called a *set simulation* of (1) with respect to possible initial states $x(0) - \hat{x}(0) \in \mathcal{X}(0)$.

The *reachable set* of (1) at time t is defined as

$$\mathcal{R}(t) \doteq \left\{ x = f^{(t)}\left(x(0)\right), x(0) \in \mathcal{X}(0) \right\}, \qquad t = 1, 2, \ldots$$

where $f^{(t)}(\cdot)$ denotes the t-stages composition $f \circ f \cdots \circ f$. Notice that in the particular case when the system (1) is actually *linear*, and $\mathcal{X}(0)$ is the hypercube $\{\|x\|_\infty \leq 1\}$, then the set $\mathcal{R}(t)$ is a convex polytope, [1]. However, in the general quadratic case $\mathcal{R}(t)$ is a complicated nonconvex set, and an exact propagation in time of $\mathcal{R}(t)$ is numerically unfeasible. An alternative to the exact propagation of the set $\mathcal{R}(t)$ is, therefore, to recursively bound $\mathcal{R}(t)$ with computationally tractable sets $\mathcal{X}(t)$ having simpler description, such as ellipsoids or orthotopes. In this latter approach, which is the one pursued in this paper, we trade some accuracy in the description of the reachable set, and possibly introduce conservatism, to gain numerical tractability and efficiency in the computations.

Denote with $\nu(k) \doteq x(k) - \hat{x}(k)$ the deviation of the actual state trajectory from the nominal one. The deviation obeys to the following time-varying quadratic difference equations:

$$\nu_i(k+1) = a_i^T(k)\nu(k) + \nu^T(k) H_i \nu(k), \qquad i = 1, \ldots, n \tag{3}$$

where, for $i = 1, \ldots, n$, and $r, c = 1, \ldots, n$

$$a_i^T(k) \doteq \left[\frac{\partial f_i}{\partial x_1} \cdots \frac{\partial f_i}{\partial x_n} \right]_{x=\hat{x}(k)};$$

$$[H_i]_{r,c} \doteq \left. \frac{\partial^2 f_i}{\partial x_r \partial x_c} \right|_{x=\hat{x}(k)}.$$

Remark 1: For the quadratic system (1), (2), we clearly have $a_i^T(k) = 2b_i^T + 2\hat{x}^T(k)Q_i$, and $H_i = Q_i$. For more general, not necessarily quadratic systems, the dynamic equations (3) may be still assumed to hold, in an approximate sense, as a second-order truncation of the Taylor series expansions of $f_i(x)$ around $\hat{x}(k)$, $i = 1, \ldots, n$. In this case, also the matrices H_i will be dependent on the time k. $\star$

In the next sections, we present our main results for recursive set simulations for the quadratic dynamics of the deviation from nominal trajectory expressed in (3).

II. Orthotopic Simulations for Quadratic Systems

In this section, we discuss the set simulation problem for the quadratic system (3), using orthotopes as bounding sets for the system state. Orthotopes permit to express the uncertainty in the initial state in the form of independent intervals. The resulting set simulation also directly provides (deterministic) intervals of confidence for each component of the system state. Assume that at a given time k it is known that $\nu(k) \in \mathcal{X}(k)$, with

$$\mathcal{X}(k) = \{\nu : \nu = \varsigma(k) + E(k)z, \ \|z\|_\infty \leq 1\}$$

where $\varsigma(k) \in \mathbb{R}^n$ describes the center of the orthotope, while $E(k) \doteq \mathrm{diag}(e_1(k), \ldots, e_n(k))$, $e_i(k) \geq 0$, $i = 1, \ldots, n$, describes the half-widths of the intervals around the center.

Considering equations (3), and substituting $\nu(k) = \varsigma(k) + E(k)z$, we obtain

$$\begin{aligned} \nu_i(k+1) = {} & a_i^T(k)\varsigma(k) + \varsigma^T(k) H_i \varsigma(k) \\ & + \left(a_i^T(k) + 2\varsigma^T(k) H_i \right) E(k) z + z^T E^T(k) H_i E(k) z \end{aligned} \tag{4}$$

Manuscript received August 5, 2002; revised November 13, 2002. Recommended by Associate Editor V. Balakrishnan.

The author is with the Dipartimento di Automatica e Informatica, Politecnico di Torino, 24-10129 Torino, Italy (e-mail: giuseppe.calafiore@polito.it).

Digital Object Identifier 10.1109/TAC.2003.811257

for $\|z\|_\infty \leq 1$, $i = 1, \ldots, n$.

Our goal is to determine a new orthotope $\mathcal{X}(k+1)$, with center $\varsigma(k+1)$ and minimal interval widths $E(k+1) = \mathrm{diag}(e_1(k+1), \ldots, e_n(k+1))$, such that $\nu(k) \in \mathcal{X}(k)$ implies that $\nu(k+1) \in \mathcal{X}(k+1)$. The problem amounts therefore to determining a minimal interval containing $\nu_i(k+1)$, for z that ranges inside the unit box $\|z\|_\infty \leq 1$. Since (4) is a quadratic (and in general nonconvex) function of z, determining the extreme values of $\nu_i(k+1)$ over the unit box is NP-hard, [18]. In the following, we propose a semidefinite relaxation of the problem, which provides a suboptimal solution that can be computed with great numerical efficiency via convex semidefinite optimization (SDP) [24]. This relaxation is then compared with a standard one based on interval arithmetics. We first state the following technical lemma.

Lemma 1 (Semidefinite Relaxation): Consider the quadratic function

$$g(z) = 2d^T z + z^T W z = \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} W & d \\ d^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}$$

where $W = W^T \in \mathbb{R}^n$, and let $g_{\max} \doteq \max_{\|z\|_\infty \leq 1} g(z)$, $g_{\min} \doteq \min_{\|z\|_\infty \leq 1} g(z)$. Then, a maximized lower bound $g^- \leq g_{\min}$ and a minimized upper bound $g^+ \geq g_{\max}$ may be computed solving the following two semidefinite (convex) programming problems in the variables $\gamma, \tau_1, \ldots, \tau_n$:

$$g^- = -\arg\min \gamma \text{ subject to:} \tag{5}$$

$$\begin{bmatrix} \mathrm{diag}(\tau_1, \ldots, \tau_n) + W & d \\ d^T & \gamma - \sum_{i=1}^n \tau_i \end{bmatrix} \succeq 0 \tag{6}$$

$$\tau_1, \ldots, \tau_n \geq 0; \tag{7}$$

$$g^+ = \arg\min \gamma \text{ subject to:} \tag{8}$$

$$\begin{bmatrix} \mathrm{diag}(\tau_1, \ldots, \tau_n) - W & d \\ d^T & \gamma - \sum_{i=1}^n \tau_i \end{bmatrix} \succeq 0 \tag{9}$$

$$\tau_1, \ldots, \tau_n \geq 0. \tag{10}$$

$\star$

Proof: We present a complete proof for the computation of the upper bound g^+; the proof for the lower bound follows from an identical reasoning. First, we observe that

$$g_{\max} = \arg\min \gamma, \text{ subject to:} \tag{11}$$

$$g(z) \leq \gamma, \ \forall z : \|z\|_\infty \leq 1. \tag{12}$$

A sufficient condition for the quadratic inequality $g(z) - \gamma \leq 0$ to hold for all z such that $z_i^2 - 1 \leq 0$, $i = 1, \ldots, n$, is given by the $\mathcal{S}$ procedure [4]. There exist nonnegative scalars $\tau_1, \ldots, \tau_n$ such that $-g(z) + \gamma + \sum_{i=1}^n \tau_i (z_i^2 - 1) \geq 0$, $\forall\, z$. Substituting $g(z) = 2d^T z + z^T W z$, and changing variable $z = -\zeta$, the previous condition is rewritten in matrix form as: $\exists \tau_1, \ldots, \tau_n \geq 0$ such that

$$\begin{bmatrix} \zeta \\ 1 \end{bmatrix}^T \begin{bmatrix} -W + \mathrm{diag}\,(\tau_1, \ldots, \tau_n) & d \\ d^T & \gamma - \sum_{i=1}^n \tau_i \end{bmatrix} \begin{bmatrix} \zeta \\ 1 \end{bmatrix} \geq 0$$

is satisfied for all ζ. Clearly, this condition holds if and only if

$$\begin{bmatrix} -W + \mathrm{diag}(\tau_1, \ldots, \tau_n) & d \\ d^T & \gamma - \sum_{i=1}^n \tau_i \end{bmatrix} \succeq 0, \qquad \tau_1, \ldots, \tau_n \geq 0 \tag{13}$$

which is a linear matrix inequality (LMI) in the variables $\gamma, \tau_1, \ldots, \tau_n$. Since (13) is a sufficient condition for the constraint (12) to hold, the feasible set implied by condition (13) is contained in the feasible set implied by condition (12) [i.e., the constraint (13) is more stringent than (12)]. Therefore, the convex semidefinite optimization problem

$$g^+ = \arg\min \gamma, \text{ subject to: } (13) \tag{14}$$

has an optimal objective value g^+ which is not smaller than $g_{\max}$, that is $g^+ \geq g_{\max}$. □

Remark 2: We remark that the upper and lower bounds obtained by means of semidefinite relaxations are among the best possible computationally efficient approximations of the actual optimal values $g_{\min}$, $g_{\max}$. For a precise assessment on the sharpness of these relaxations, and for further recent results, the interested reader is referred to [10], [16], [18], and [19].

We also notice that, although the focus of this note is on the case when $g(z)$ is a *quadratic* function, the same approach could in principle be applied to generic polynomial systems, using the techniques recently developed in [16] and [21]. $\star$

Alternative bounds on $g_{\min}$, $g_{\max}$ may be computed using standard interval arithmetics, assuming that the monomials appearing in the quadratic forms are all independent, see, for instance, [12]. These bounds are computationally cheaper to determine compared to those obtained by means of semidefinite relaxations, but are significantly looser. This introduces undesirable additional conservatism in the simulations, as further discussed in the examples that follow.

The computation of bounds on $g(z)$ using the independent monomials relaxation is stated in the following lemma, whose simple proof is omitted.

Lemma 2 (Independent Monomials Relaxation): Let all symbols be defined as in Lemma 1. Then, $[g_{\min}, g_{\max}] \subseteq [g^{I-}, g^{I+}]$, with

$$g^{I-} = -2\sum_{i=1}^n |d_i| + \sum_{i=1}^n \sigma_-(w_{ii}) - 2\sum_{\substack{i,j=1 \\ j>i}}^n |w_{ij}|$$

$$g^{I+} = 2\sum_{i=1}^n |d_i| + \sum_{i=1}^n \sigma_+(w_{ii}) + 2\sum_{\substack{i,j=1 \\ j>i}}^n |w_{ij}|$$

where $\sigma_+(x) = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$, $\sigma_-(x) = \begin{cases} x, & \text{if } x < 0 \\ 0, & \text{otherwise} \end{cases}$ and w_{ij} denotes the element in the ith row and jth column of W. $\star$

The next theorem reports our main result for recursive orthotopic simulation of quadratic systems.

Theorem 1 (Orthotopic Simulation): Consider (3). Let $e_i(k) \geq 0$, $i = 1, \ldots, n$, and $\varsigma(k) \in \mathbb{R}^n$ be given, and define $E(k) \doteq \mathrm{diag}(e_1(k), \ldots, e_n(k))$. Let further, for $i = 1, \ldots, n$

$$d^{(i)T}(k) \doteq \left(a_i^T(k) + 2\varsigma^T(k) H_i\right) E(k) \in \mathbb{R}^{1,n}$$

$$W^{(i)}(k) \doteq E^T(k) H_i E(k) \in \mathbb{R}^{n,n}$$

and let $g_i^-(k)$, $g_i^+(k)$ be optimized bounds on the quadratic function $g^{(i)}(z) \doteq 2d^{(i)T}(k)z + z^T W^{(i)}(k) z$, computed according to Lemma 1. Define

$$\nu_i^-(k+1) \doteq a_i^T(k)\varsigma(k) + \varsigma^T(k) H_i \varsigma(k) + g_i^-(k) \tag{15}$$

$$\nu_i^+(k+1) \doteq a_i^T(k)\varsigma(k) + \varsigma^T(k) H_i \varsigma(k) + g_i^+(k). \tag{16}$$

Then, the orthotope $\mathcal{X}(k+1)$ with center

$$\varsigma(k+1) = \left[\frac{\nu_1^+(k+1) + \nu_1^-(k+1)}{2} \cdots \frac{\nu_n^+(k+1) + \nu_n^-(k+1)}{2}\right]^T$$

and half-widths $e_i(k+1) = ((\nu_i^+(k+1) - \nu_i^-(k+1))/2)$, $i = 1, \ldots, n$, contains the state $\nu(k+1)$, for any $\nu(k) \in \mathcal{X}(k)$. $\star$

Proof: The proof is immediate, substituting the linear and quadratic terms in (4) with the bounds $g_i^-(k)$, $g_i^+(k)$ determined according to Lemma 1. □

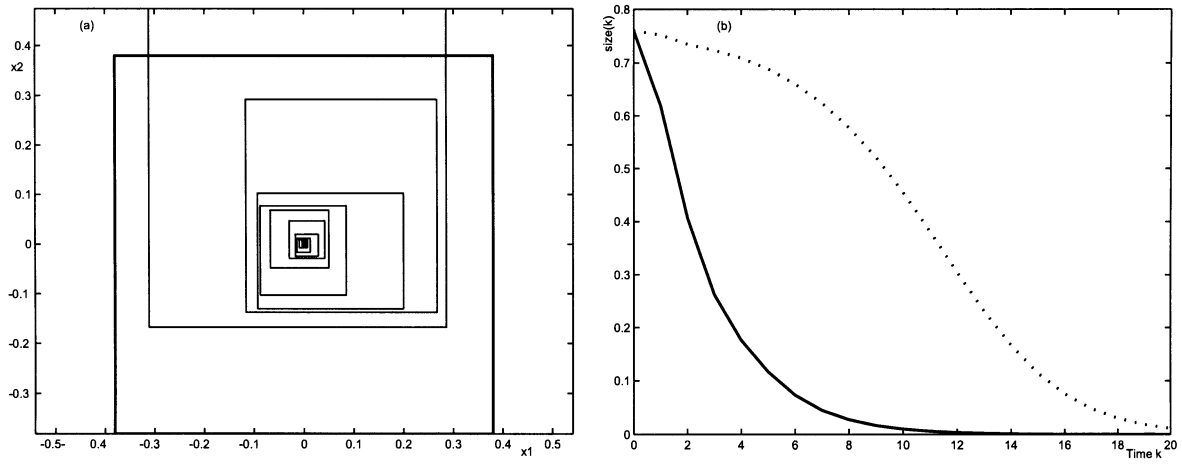

Fig. 1. Set simulation for Example 2.1, with $e_1(0) = e_2(0) = 0.38$. (a) Sequence of rectangles $\mathcal{X}(k)$: the bold line shows the initial rectangle $\mathcal{X}(0)$. (b) Size of optimal rectangles (sum of half-widths) versus time: the solid line shows the size for the simulation obtained via semidefinite relaxation, while the dotted line shows the size obtained via independent monomials relaxation.

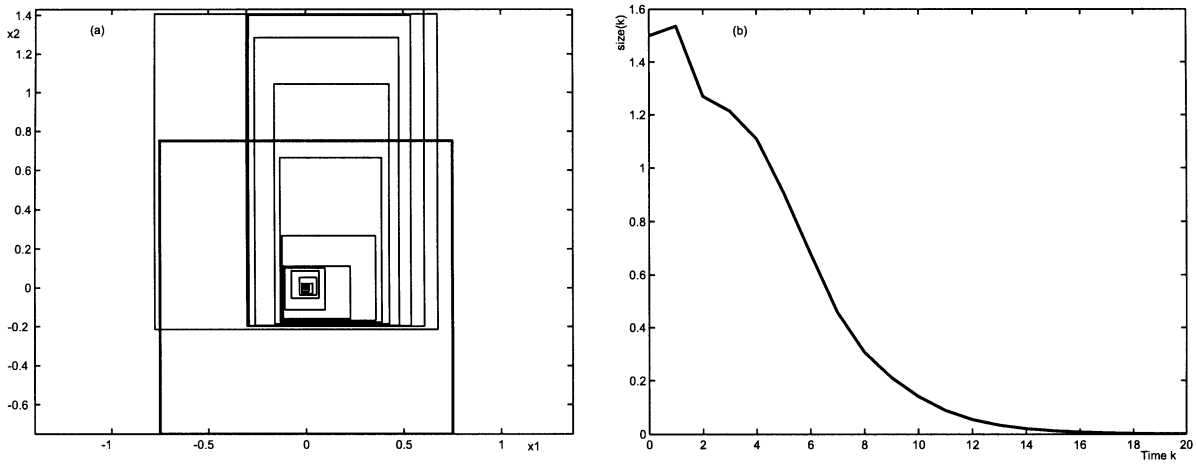

Fig. 2. Set simulation for Example 2.1, with $e_1(0) = e_2(0) = 0.75$. (a) Sequence of rectangles $\mathcal{X}(k)$ obtained via semidefinite relaxations: the bold line shows the initial rectangle $\mathcal{X}(0)$. (b) Size of optimal rectangles (sum of half-widths) versus time.

Remark 3: Applying Theorem 1 recursively for $k = 0, 1, \ldots$, starting from an initial orthotope $\mathcal{X}(0)$, we obtain an orthotopic simulation for the quadratic system (3). In the aforementioned theorem, one may alternatively use the coarser bounds derived in Lemma 2 instead of the tight bounds $g_i^-(k), g_i^+(k)$ derived in Lemma 1. This would speed up the simulation (since no optimization is required at each step), at the expense of possibly severe conservatism, as shown in the following. We notice in particular that the computational effort required to solve each of the semidefinite programs in Lemma 1 to a given accuracy (using a general-purpose SDP solver and not exploiting structure) grows with problem size as $O(n^{1/2})O(n^4)$ in the worst case; see [24], while computing the bounds in Lemma 2 basically requires $n(n+3)/2$ additions. $\star$

Remark 4: We remark that set simulation can be a fast and effective tool to analyze the domain of attraction (DA) around an equilibrium: if $\hat{x}(0)$ is an equilibrium point for (1), we obtain time-invariant dynamics (3) describing the deviations from equilibrium. The convergence to zero of the orthotopic simulation on (3) then *proves* that any trajectory originating in $\mathcal{X}(0)$ converges asymptotically to the origin, thus eliminating the need of resorting to the Lyapunov approach. However, we remark that other specific techniques have been developed for this purpose. For instance, when a quadratic Lyapunov function is given in advance, a method for determining an ellipsoidal region of attraction for quadratic, continuous-time systems has been developed in [23], while a technique for constructing optimal quadratic Lyapunov functions for generic polynomial systems has been recently proposed in [6]. $\star$

Example 2.1: Consider the quadratic system

$$x_1(k+1) = 0.5x_1^2(k) - 0.5x_2^2(k) + 0.4x_1(k)x_2(k) + 0.6x_2(k)$$
$$x_2(k+1) = 0.6x_1^2(k) + 0.5x_2^2(k) + 0.6x_1(k)x_2(k) - 0.6x_1(k)$$

for which $\hat{x}(0) = 0$ is an equilibrium point. Considering an initial rectangle with $e_1(0) = e_2(0) = 0.38$ and $\varsigma(0) = 0$, we obtain the orthotopic simulation in Fig. 1. The sequence of rectangles produced by the set simulation is depicted in Fig. 1(a), while Fig. 1(b) shows the convergence of the size of $\mathcal{X}(k)$ (sum of half-widths), which proves that the equilibrium is attractive,[1] and that all points in $\mathcal{X}(0)$ belong to the domain of attraction relative to the origin.

From Fig. 1(b), we notice the improvement in size reduction obtained by the proposed semidefinite relaxation approach over the standard interval approach. Indeed, enlarging the region $\mathcal{X}(0)$ to $e_1(0) = e_2(0) = 0.75$, using the semidefinite relaxation method we obtain the (convergent) set simulation in Fig. 2, while the simulation based on the independent monomials relaxation fails to converge for $e_1(0) = e_2(0) \geq 0.39$.

[1]In this case, the asymptotic stability of the equilibrium is also easily proved by means of Lyapunov linearization method.

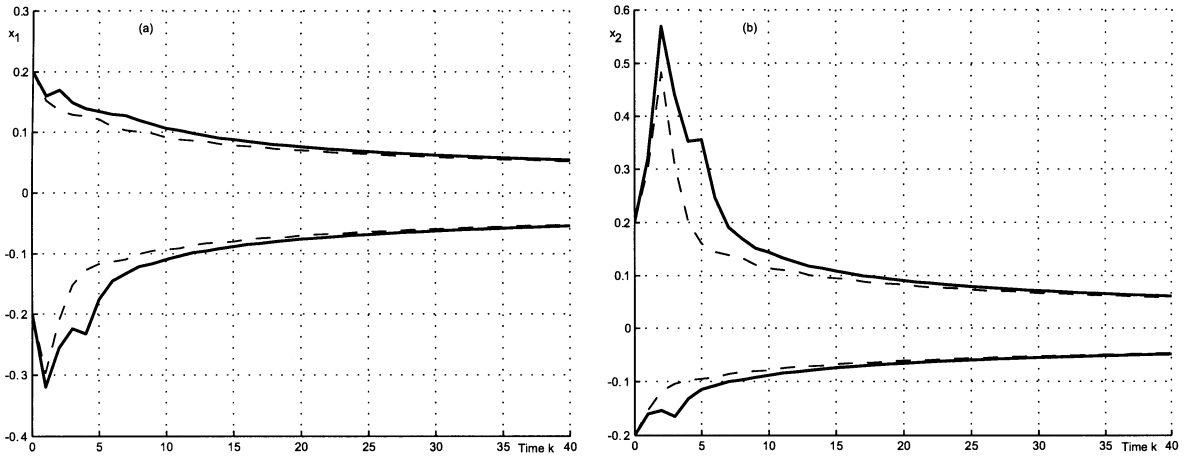

Fig. 3. Set simulation for Example 2.2. (a) Solid line: bounds on state x_1, dotted line: envelope of Monte-Carlo simulations. (b) Solid line: bounds on state x_2, dotted line: envelope of Monte-Carlo simulations.

Example 2.2: As a further example, consider the third-order quadratic system

$$\begin{aligned} x_1(k+1) &= x_2(k) - 2x_2^2(k) + 0.5x_1(k)x_3(k) \\ x_2(k+1) &= -x_1(k) + 2x_1^2(k) + 0.5x_2(k)x_3(k) \\ x_3(k+1) &= 0.5x_3(k) + x_1(k)x_3(k). \end{aligned} \tag{17}$$

For this system, the linearization approach does not provide information about the stability of the equilibrium $\hat{x}(0) = 0$. However, the orthotopic simulation with $e_1(0) = e_2(0) = e_3(0) = 0.2$, $\varsigma(0) = 0$ shows that any trajectory originating in $\mathcal{X}(0)$ remains bounded, and gives additional insight on the qualitative behavior of the response. Fig. 3 shows the bounds obtained on the first two states using the proposed semidefinite relaxations, compared with the envelope of the trajectories obtained running 1000 Monte Carlo simulations on the system. The Monte Carlo simulations have been performed generating uniformly random initial states in $\mathcal{X}(0)$. We remark that the set simulation using the independent monomials relaxation of Lemma 2 fails again to converge in this case, contrary to that based on semidefinite relaxation.

III. Ellipsoidal Simulation for Quadratic Systems

Numerically efficient set simulations for quadratic systems may also be obtained using ellipsoids as bounding sets for the system state. Ellipsoidal sets provide a good tradeoff between conservatism of the simulation and corresponding numerical complexity, as discussed in the following. They are widely used in state estimation and filtering problems with unknown-but-bounded noise, see the classical references [15], [20], [22], and the literature cited therein.

Consider the deviations equations (3), and assume that at a given time k it is known that $\nu(k) \in \mathcal{X}(k)$, with

$$\mathcal{X}(k) = \{\nu : \nu = E(k)z,\ \|z\| \le 1\}$$

where $E(k) \in \mathbb{R}^{n,n}$ is a given matrix which describes the shape of the ellipsoid, and $|\cdot|$ denotes the standard Euclidean norm. For simplicity in the derivation, we here consider that all ellipsoids are centered in the origin; the case in which the centers $\varsigma(k)$ are considered follows easily from the same reasoning; see Remark 5.

Considering again (3), we rewrite this syetm in vector form as

$$\nu(k+1) = A(k)E(k)z + b(k) \tag{18}$$

where $A(k) \doteq [a_1(k) \cdots a_n(k)]^T$, $b_i(k) \doteq z^T E^T(k) H_i E(k) z, i = 1, \ldots, n$. Our goal is to determine a minimal ellipsoid $\mathcal{X}(k+1)$, with shape matrix $E(k+1)$, such that $\nu(k) \in \mathcal{X}(k)$ implies that $\nu(k+1) \in \mathcal{X}(k+1)$. As a geometrical measure of the "size" of $\mathcal{X}(k+1)$ we adopt the sum of the squared semi-axes lengths of the ellipsoid, which is given by $\mathrm{Tr}P(k+1)$, with $P(k+1) = E(k+1)E^T(k+1)$.

First, we notice that $\|z\| \le 1$ implies that $|b_i(k)| \le \|E^T(k)H_iE(k)\| \doteq m_i(k)$, $i = 1, \ldots, n$, therefore, $\nu(k+1)$ belongs to the set

$$\mathcal{W}(k+1) \doteq \left\{\nu : \nu = A(k)E(k)z + \sum_{i=1}^{n} m_i(k) 1_i u_i, \|z\| \le 1, |u_i| \le 1, \qquad i = 1, \ldots, n\right\} \tag{19}$$

where $1_i \in \mathbb{R}^n$ has all zero components, except the ith, which is equal to one. The set $\mathcal{W}(k+1)$ is the Minkowski sum of $n+1$ bounded and possibly flat ellipsoids centered in the origin. Our second step amounts then to determining a minimal ellipsoid that contains $\mathcal{W}(k+1)$.

Notice that determining the *optimal* minimum trace ellipsoid that contains $\mathcal{W}(k+1)$ (this ellipsoid exists and it is unique, by an extension of the John theorem, [13]) is a numerically hard problem. Indeed, this problem has been shown to be equivalent to the maximization of a convex quadratic form under convex quadratic constraints, which is known to be NP-hard, [11], [18]. For these reasons, it is a common approach to pursue suboptimality, looking for the minimum trace ellipsoid among a parameterized family of ellipsoids. The following lemma reports a version of a well-known result due to Schweppe [22] which gives a parameterization of a family of ellipsoids that contain the sum of K given ellipsoids.

Lemma 3 (Schweppe): Let

$$\mathcal{E}_i \doteq \{x \in \mathbb{R}^n : x = c_i + E_i z_i, \|z_i\| \le 1\}, \qquad i = 1, \ldots, K$$

be K given ellipsoids with center c_i and shape matrix $E_i \in \mathbb{R}^{n, n_{z_i}}$. Denote with $P_i \doteq E_i E_i^T$ the squared shape matrices of the ellipsoids. Then, for any $\tau_i > 0$, $i = 1, \ldots, K$ such that $\sum_{i=1}^K \tau_i \le 1$, the ellipsoid $\mathcal{E}(K) \doteq \{x \in \mathbb{R}^n : x = c(K) + E(K)z, \|z\| \le 1\}$ with $c(K) \doteq \sum_{i=1}^K c_i$, $P(K) \succeq \sum_{i=1}^K 1/\tau_i P_i$, being $P(K) \doteq E(K)E^T(K)$, contains the Minkowski sum $\mathcal{S}(K) \doteq \sum_{i=1}^K \mathcal{E}_i$. ⋆

The minimum trace ellipsoid in the above family may be computed in closed form as detailed next; see [7].

Lemma 4: Let all symbols be defined as in Lemma 3. The minimum trace ellipsoid $\mathcal{E}(K)$ in the Schweppe parametric family containing $\mathcal{S}(K) = \sum_{i=1}^K \mathcal{E}_i$ is obtained for

$$\begin{aligned} \tau_i &= \frac{s_i}{\sum_{j=1}^K s_j}, \qquad i = 1, \ldots, K \\ s_i &\doteq \sqrt{\mathrm{Tr} E_i E_i^T}. \end{aligned}$$

Moreover the optimal size is $s(K) \doteq \sqrt{\mathrm{Tr}P(K)} = \sum_{i=1}^K s_i$. ⋆

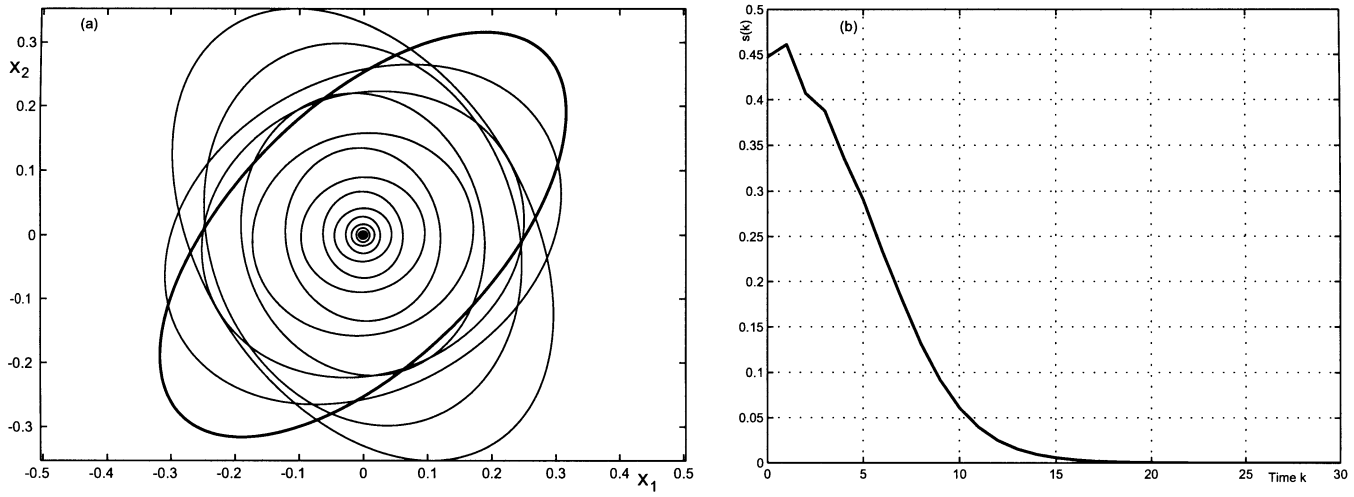

Fig. 4. (a) Sequence of ellipsoids $\mathcal{X}(k)$ of Example 3.1; the bold line shows the initial ellipsoid $\mathcal{X}(0)$. (b) Size of optimal ellipsoids versus time.

As previously discussed, the ellipsoid computed in Lemma 4 is the optimal one within the considered parametric family, but it is not in general guaranteed to be optimal among all possible ellipsoids. However, recent results on semidefinite relaxations permit to state precise bounds on the quality of this suboptimal solution. The following bound, which we report without proof, is a consequence of a result given in [19]. A similar bound for minimum *volume* ellipsoids covering the sum of ellipsoids $\mathcal{S}(K)$ has been stated in [2].

Lemma 5: Let all symbols be defined as in Lemma 3 and Lemma 4. Let $P(K)$ be the squared shape matrix of the (sub-optimal) minimum trace ellipsoid $\mathcal{E}(K)$ containing $\mathcal{S}(K) = \sum_{i=1}^{K} \mathcal{E}_i$, computed according to Lemma 4. Let $P_*(K)$ be the squared shape matrix of the actual minimum trace ellipsoid $\mathcal{E}_*(K)$ containing $\mathcal{S}(K) = \sum_{i=1}^{K} \mathcal{E}_i$, computed among *all possible* ellipsoids. Then, it holds that

$$\frac{\sqrt{\mathrm{Tr}P(K)}}{\sqrt{\mathrm{Tr}P_*(K)}} \leq \sqrt{\frac{\pi}{2}} \simeq 1.25. \qquad \star$$

Notice that this result states that the size of the suboptimal bounding ellipsoid computed by means of Lemma 4 is at most 25.3% larger that the actual optimal size, and that this figure holds independently of the space dimension n and the number K of ellipsoids to be approximated.

Returning our attention to the quadratic system (3), applying Lemma 4 to (19) we obtain the following result for robust ellipsoidal simulation.

Theorem 2 (Ellipsoidal Simulation): Consider the system (3). Let $E(0) \in \mathbb{R}^{n,n}$ be a given matrix, and $\mathcal{X}(0) \doteq \{\nu : \nu = E(0)z, \|z\| \leq 1\}$. Further, let $E(k)E^T(k) = P(k)$, $1_i \in \mathbb{R}^n$ with all zero entries, except the ith, which is equal to one, and $m_i(k) \doteq \|E^T(k)H_iE(k)\|$, $i = 1, \ldots, n$. Then, the recursion

$$s(k+1) = \sqrt{\mathrm{Tr}A(k)P(k)A^T(k)} + \sum_{i=1}^{n} m_i(k)$$

$$P(k+1) = s(k+1)\left(\frac{A(k)P(k)A^T(k)}{\sqrt{\mathrm{Tr}A(k)P(k)A^T(k)}} + \sum_{i=1}^{n} m_i(k)1_i 1_i^T\right)$$

for $k = 0, 1, \ldots$, determines a suboptimal sequence of sets

$$\mathcal{X}(k) \doteq \{\nu : \nu = E(k)z, \|z\| \leq 1\}, \qquad k = 0, 1, \ldots$$

having size $\sqrt{\mathrm{Tr}P(k)} = s(k)$, such that $\nu(k) \in \mathcal{X}(k)$, $k = 0, 1, \ldots$, for any initial condition $\nu(0) \in \mathcal{X}(0)$. $\star$

Remark 5: A similar result can be stated for the case when also the ellipsoids centers $\varsigma(k)$ are considered in the simulation, i.e., when we assume $\mathcal{X}(k) \doteq \{\nu : \nu = \varsigma(k) + E(k)z, \|z\| \leq 1\}$. It can be readily proven in this case that if the centers follow the recursion

$$\varsigma(k+1) = A(k)\varsigma(k) + \left[\varsigma^T(k)H_1\varsigma(k) \cdots \varsigma^T(k)H_n\varsigma(k)\right]^T$$

for $k = 0, 1, \ldots$, then the translated deviations $\tilde{\nu}(k) \doteq \nu(k) - \varsigma(k)$ are such that

$$\tilde{\nu}(k+1) = \left(A(k) + 2\begin{bmatrix}\varsigma^T(k)H_1 \\ \vdots \\ \varsigma^T(k)H_n\end{bmatrix}\right) E(k)z + b(k).$$

Comparing this latter expression to (18), we can compute a zero-centered ellipsoidal simulation for $\tilde{\nu}(k)$, $k = 0, 1, \ldots$, and then offset this simulation by $\varsigma(k)$ in order to obtain $\mathcal{X}(k)$, $k = 0, 1, \ldots$. $\star$

Example 3.1: Consider again the quadratic system of Example 2.1, for which $\hat{x}(0) = 0$ is an equilibrium point. We run the set simulation of Theorem 2, starting from an initial ellipsoid $\mathcal{X}(0)$, with

$$E(0) = \begin{bmatrix}.3 & .1 \\ .1 & .3\end{bmatrix}.$$

The sequence of ellipsoids produced by the set simulation is depicted in Fig. 4(a), while Fig. 4(b) shows the convergence of $s(k)$, which proves that the equilibrium is attractive, and that all points in $\mathcal{X}(0)$ belong to the domain of attraction relative to the origin.

IV. Conclusion

In this paper, we presented computationally efficient algorithms for orthotopic and ellipsoidal set simulations for nonlinear quadratic systems. The orthotopic simulation requires the solution of n SDPs at each step (Theorem 1), while the ellipsoidal simulation only requires the implementation of an algebraic matrix recursion (Theorem 2).

One single set simulation provides information on the qualitative behavior of the system response for a whole set of initial conditions. The convergence of the set simulation is a (readily computable) sufficient condition for the attractivity of an equilibrium, and directly provides a region contained in the domain of attraction of the system.

We remark that the proposed techniques can also be employed in the context of set-membership filtering for nonlinear systems: the set simulation up to time T constitutes indeed the prediction step of the filter, and yields an outer bounding set for the achievable states, which is then to be updated with the upcoming measurement. We also notice that

further refinements on the relaxation introduced in Lemma 1 may be obtained using higher order semidefinite relaxations (see, for instance, [16] and [21, Ch. 6]). Higher order relaxations potentially reduce the conservatism of the interval bounds, at the expense of increased complexity of the computations. These latter techniques may also be employed to extend the methodology introduced in this note to generic polynomial systems.

REFERENCES

[1] B. R. Barmish and J. Sankaran, "The propagation of parametric uncertainty via polytopes," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 346–349, Apr. 1979.

[2] A. Ben-Tal and A. Nemirovski, *On Ellipsoidal Approximations of Sums of Ellipsoids*. Philadelphia, PA: SIAM, 2001.

[3] D. P. Bertsekas and I. B. Rhodes, "Recursive state estimation with a set-membership description of the uncertainty," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 117–128, Feb. 1971.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Philadelphia, PA: SIAM, June 1994, vol. 15.

[5] A. C. Brooms, B. Kouvaritakis, and Y. I. Lee, "Constrained MPC for uncertain linear systems with ellipsoidal target sets," *Syst. Control Letters*, vol. 44, pp. 157–166, 2001.

[6] G. Chesi, A. Tesi, and A. Vicino, "Computing optimal quadratic Lyapunov functions for polynomial nonlinear systems via LMIs," presented at the 15th IFAC World Congress, Barcelona, Spain, July 2002.

[7] C. Durieu, E. Walter, and B. Polyak, "Trace versus determinant in ellipsoidal outer-bounding with application to state estimation," in *Proc. 13th IFAC World Congress*, San Francisco, CA, 1996, pp. 43–48.

[8] L. El Ghaoui and G. Calafiore, "Worst-case simulation of uncertain systems," in *Lecture Notes in Control and Information Sciences*. London, U.K.: Springer-Verlag, 1999, vol. 245, pp. 134–146.

[9] ——, "Robust filtering for discrete-time systems with bounded noise and parametric uncertainty," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1084–1089, July 2001.

[10] M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," *J. ACM*, vol. 42, pp. 1115–1145, 1995.

[11] D. Henrion, S. Tarbouriech, and D. Arzelier, "LMI approximations for the radius of the intersection of ellipsoids," *J.Optim. Theory Applicat.*, vol. 108, no. 1, pp. 1–28, 2001.

[12] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis*. London, U.K.: Springer-Verlag, 2001.

[13] O. N. Kiselev and B. T. Polyak, "Ellipsoidal estimation with respect to a generalized criterion," *Automat. Remote Control*, vol. 52, pp. 1281–1292, 1992.

[14] D. E. Koditschek and K. S. Narendra, "The stability of second-order quadratic differential equations," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 783–798, Aug. 1982.

[15] A. B. Kurzhanski and I. Valyi, *Ellipsoidal Calculus for Estimation and Control*. Boston, MA: Birkhauser, 1996.

[16] J. B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM J. Optim.*, vol. 11, no. 3, pp. 796–817, 2001.

[17] R. R. Mohler, *Bilinear Control Processes*. New York: Academic, 1973.

[18] A. Nemirovski, C. Roos, and T. Terlaky, "On maximization of quadratic form over intersection of ellipsoids with common center," *Math. Program.*, vol. 86, no. 3, pp. 463–473, 1999.

[19] Y.Yu Nesterov, "Semidefinite relaxation and nonconvex quadratic optimization," *Optim. Meth. Software*, vol. 12, pp. 1–20, 1997.

[20] J. P. Norton and S. M. Veres, "Developments in parameter bounding," in *Topics in Stochastic Systems: Modeling, Estimation and Adaptive Control*, L. Gerencser and P. E. Caines, Eds. Berlin, Germany: Springer-Verlag, 1991, pp. 1137–1158.

[21] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Inst. Technol., Pasadena, CA, 2000.

[22] F. C. Schweppe, "Recursive state estimation: unknown but bounded errors and system inputs," *IEEE Trans. Automat. Contr.*, vol. AC-13, pp. 22–28, 1968.

[23] A. Tesi, F. Villoresi, and R. Genesio, "On the stability domain estimation via a quadratic Lyapunov function: convexity and optimality properties for polynomial systems," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1650–1656, Nov. 1996.

[24] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49–95, March 1996.

Decentralized Control Design for Large-Scale Systems With Strong Interconnections Using Neural Networks

Sunan Huang, Kok Kiong Tan, and Tong Heng Lee

***Abstract*—We propose a decentralized neural network (NN) controller for a class of large-scale nonlinear systems with the strong interconnections. The NNs are used to approximate the unknown subsystems and interconnections. Due to the functional approximation capabilities of NNs, the additional precautions are not required to be made for avoiding the possible control singularity problems. Semiglobal asymptotic stability results are obtained and the tracking error converges to zero. Furthermore, the issue of transient performance of the subsystems is also addressed under an analytical framework.**

***Index Terms*—Adaptive control, decentralized control, neural networks.**

I. INTRODUCTION

In classical control theory, it is assumed that control actions are undertaken by a single controller that has all the available information about the system. While there are obvious theoretical advantages, control centralization may be difficult for a number of economic and technical reasons. In recent years, there has been an increased interest in the development theories for large-scale systems (see [1]–[4]). A significant proportion of these effects deals with the problem of control design methods implemented in a decentralized way. Earlier versions of the decentralized adaptive control methods were focused on control of large-scale linear systems. However, most physical systems are inherently nonlinear. Research on decentralized control for nonlinear systems was carried out in [5]–[11] and [13]. These previous works consider subsystems which are linear in a set of unknown parameters [5]–[7], [11], or consider the isolated subsystems to be known [9], [10], [13], such as input gain functions. Recently, Spooner and Passino [25] proposed a radial basis neural network control method to approximate unknown functions in nonlinear subsystems which may not be linearly parameterized. Specifically, the direct NN controller in [25], requires neither the knowledge of input gain functions, nor the direct estimation of the unknown input gain function. It thus avoids the control singularity problem. On the other hand, the uncertainty also may appear in the interconnections. Most of the literature on decentralized control is focused on systems with first-order interconnections [1]–[8]. These results cannot guarantee stability when the interconnections are of higher order [9]. The results on decentralized control of large-scale systems with higher order interconnections are by [9], [11]. In [12] and [13], the uncertain interconnections are extended to be bounded by known nonlinear functions.

Manuscript received November 25, 2001; revised September 16, 2002. Recommended by Associate Editor H. Wang.

The authors are with Department of Electrical and Computer Engineering, National University of Singapore, Singapore (e-mail: elehsn@nus.edu.sg).

Digital Object Identifier 10.1109/TAC.2003.811258